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ON RELATIVISTIC ANALOGUES OF PARTICLE DYNAMICS*

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The construction of the main characteristics of all the theoretically admissible generalizations of the classical mechanics of one-dimensional motions of a particle is discussed. These characteristics are the transformations which connect the inertial frames of reference, the variation factors of the geometric scales and the clock behaviour in the systems, the laws of velocity transformation, and the laws of dynamics. The most interesting result is to discover a mechanics which is infinitely close to relativistic mechanics yet differs from it in principle. The difference consists in violating the parity (scales and clocks which move in exactly opposite directions with the same velocity behave differently).

The particle inertial motions, which are monitored (mainly) by a conformal group, and which have for long played a special role in physics, are constructed and studied. The problem is solved in the complete statement. It turns out to be necessary here for the order of the equations to be the third (and not the second) and for motions with velocity higher than that of light to be possible.

In the Ehrlang program /1/, Klein stated a general idea according to which any geometry is a theory of invariants of a transformation group. In physics, this approach is embodied in variance principles /2/. The generality of the latter is expressed in particular by the fact that, not only the Galilean and the Lorentz, but also any other transformation group of space-time can be associated with its own relativity and its own mechanics. This point of view is worth using to construct a variety of mechanical models.

We shall first make some general remarks.

Transformation groups. For the purpose of theoretical physics, studies have long been made into the theory of compressions and deformations of Lie groups and algebras /3-6/. Groups of dimensionality 10 are the most interesting. But the problem of describing them is so complicated that so far only particular (albeit important) results have been obtained, see e.g., /7, 8/. One-dimensional mechanics are controlled by three-parameter groups. To construct these mechanics, it is by no means sufficient simply to list the groups, which are traditionally restricted by special guides /9/. We gave the requisite fuller description of three-dimensional Lie algebras in space of structural constants in /10/. It enabled us to indicate the passages to the limit between algebras and to construct in the simplest way all the coordinate realizations of the respective transformation groups. This description will also be used in the present paper. Discrete components, in particular, reflections, are not included in the groups.

Inertial reference systems. We take the following informal definitions which are sufficient for our purposes below.

The reference system $\{x, t\}$ is a physical body which has its own clock at every point. Points of the body are arithmetized by the variable x, and clock readings by the variable t.

Inertial frames of reference are systems which transform from one to another given transformation group $G: x' = \varphi(x, t, \tau),\ t' = \psi(x, t, \tau)$. This last definition obviously depends only on the group structure.

Any given group G will obviously transform some family of motions into itself. This family is the wider, the great the number of initial conditions that can be present in it. The minimal families of this kind, in which the dependence of the law of motion on the initial conditions contains the least possible arbitrariness, are the most interesting. Here and below, such particle motions are called inertial motions.

A moving reference body is obviously formed by points which are at rest relative to the body. Consequently, the points of a moving body move with respect to a fixed point according to the law $\varphi(x, t, \tau) = \text{const}$. This family of motions is preserved by the groups. It is minimal and therefore describes the inertial motions.

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In the first part of the paper we consider linear transformations. The reference body is therefore a rigid body which moves relative to another rectilinearly and uniformly.

Invariance of the dynamic law. Force in dynamics is a fundamental physical quantity, like time, extension, and mass. For force, as for the other quantities, we have to indicate the transformation law when passing from one inertial frame of reference to another (thus, in classical particle dynamics, components of a force are preserved under Galilean motions and trnslations, and are transformed like particle coordinates in the case of rotations). This transforms the kinematic group which controls the law of inertia into an isomorphic dynamic group. It is to transformations of the group that the dynamic law must be invariant.

In each specific case of particle interaction with the surroundings, the force is converted into a function of the coordinates, velocity, and time, and the law of dynamics, into equations of motion. These do not in general need to be invariant under the kinematic group transformations.

1. Initial relations. Corollaries of the homogeneity condition. Let x be the geometrical coordinate of the particle in an inertial frame of reference, and t the time measured by clock in the same system. It follows from our assumption that space-time is homogeneous that measurements of the physical variables x, t are invariant under a change of reference origin $x' = x + \tau_1$, $t' = t + \tau_2$, where τ_1, τ_2 are canonical group parameters. Corresponding to the written transformation we have the operators

$$X_1 = \partial/\partial x, \quad X_2 = \partial/\partial t \tag{1.1}$$

(We shall sometimes in future for convenience put $x = x_1$, $t = x_2$.). We include the operators (1.1) in the basis of the three-dimensional algebras considered below. Since $[X_1, X_2] = 0$, then

$$c_{12}{}^1 = c_{12}{}^2 = c_{12}{}^3 = 0 \tag{1.2}$$

The simple groups G^1, G^2 (motions of the Lobachevskii plane and rotations) do not satisfy conditions (1.2) and are excluded from further consideration /10/.

Components of the operator X_3. The remaining commutation relations in the three-dimensional algebras are

$$[X_i, X_3] = c_{i3}{}^j X_j, \; X_3 = \xi_1\partial/\partial x_1 + \xi_2\partial/\partial x_2$$

Here and throughout, $i, j = 1, 2$. Summation is performed with respect to repeated indices. In the scalar form

$$\partial\xi_i/\partial x_j = c_{j3}{}^i + c_{j3}{}^3\xi_i \tag{1.3}$$

The Jacobi condition is

$$c_{13}{}^3 c_{23}{}^i - c_{23}{}^3 c_{13}{}^i = 0$$

We now have to integrate relations (1.3), regarding them as equations in the unknown functions ξ_i. From the solutions obtained we have to choose those for which passage to the limit to the Galilean operator is possible:

$$X_3 \to t\partial/\partial x \equiv x_2\partial/\partial x_1 \tag{1.4}$$

With $c_{13}{}^3 = c_{23}{}^3 = 0$ the most general homogeneous solution of Eqs.(1.3), that satisfies condition (1.4), is

$$\xi_i = c_{j3}{}^i x_j, \quad c_{23}{}^1 \neq 0 \tag{1.5}$$

In the cases $(c_{13}{}^3)^2 + (c_{23}{}^3)^2 \neq 0$, corresponding to the same transformation group (G^{c_0}, $c_0 = 1$), Eqs.(1.3) also have a non-linear solution, which we shall not discuss.

The question arises as to whether the problem can be simplified by taking (1.5), not in the general form, but with fixed values of the structural constants that correspond to non-isomorphic algebras. The point is that the structural constants can be specialized in different ways within the same algebra. Different expressions are then obtained for transformations of the same group. The question is, which of them to prefer. We cannot in principle answer this question without new physical arguments. It is therefore better to keep the discussion general until the final results are obtained. This matter will be considered in Sect.4.

Groups of motions. To the groups of motions there correspond operators with components that satisfy the Killing equations. For Riemann spaces with metric $ds^2 = a_{ij}dx_i dx_j$ these equations are

$$\xi_k \frac{\partial a_{lm}}{\partial x_k} + a_{mk}\frac{\partial \xi_k}{\partial x_l} + a_{lk}\frac{\partial \xi_k}{\partial x_m} = 0$$

Here and below, $k, l, m = 1, 2, 3$.

Substitution of the components of the operators X_1 ($\xi_1{}^{(1)} = 1$, $\xi_2{}^{(1)} = 0$), X_2 ($\xi_1{}^{(2)} = 0$, $\xi_2{}^{(2)} = 1$),

$X_3\,(\xi_1^{(3)} = \xi_1,\ \xi_2^{(3)} = \xi_2)$ gives

$$a_{ij} = \text{const},\quad c_{13}{}^1 a_{11} + c_{13}{}^2 a_{12} = 0,\quad c_{23}{}^1 a_{12} + c_{23}{}^2 a_{22} = 0$$
$$c_{23}{}^1 a_{11} + (c_{13}{}^1 + c_{23}{}^2) a_{12} + c_{13}{}^2 a_{22} = 0$$

Obviously, with

$$(c_{13}{}^1 + c_{23}{}^2) D \neq 0 \quad (D = c_{13}{}^2 c_{23}{}^1 - c_{23}{}^2 c_{13}{}^1) \tag{1.6}$$

A transformation group does not preserve a metric. With

$$D = 0 \tag{1.7}$$

the preserved metric is degenerate: $ds^2 = (c_{23}{}^2 dx_1 - c_{23}{}^1 dx_2)^2$. With $c_{13}{}^1 + c_{23}{}^2 = 0$ the metric

$$ds^2 = -(c_{13}{}^2/c_{23}{}^1) dx_1{}^2 - 2\,(c_{23}{}^2/c_{23}{}^1) dx_1 dx_2 + dx_2{}^2$$

is preserved. It is indefinite for $D > 0$ and definite for $D < 0$.

Final transformations. The required transformation group of the geometric coordinate and time, which corresponds to operator X_3, is given by the solution of the Cauchy problem

$$dx_i'/d\tau = c_{j3}^{i} x_j', \quad x_i'\,|_{\tau=0} = x_i \tag{1.8}$$

The solution of Eqs.(1.8) can be written as

$$x_i' = b_{ij}(\tau) x_j, \quad b_{ij}(0) = \delta_j{}^i \tag{1.9}$$

where $\delta_j{}^i$ is the Kronecker delta. The inverse transormation is

$$x_i = b_{ij}(-\tau) x_j' \tag{1.10}$$

The canonical parameter τ is usually replaced by the constant velocity V of the motion of the body of the reference system $K' = \{x', t'\}$ relative to the body of the reference system $K = \{x, t\}$, which is taken as fixed (Fig.1). The replacement is made in accordance with the equation

$$b_{11}(\tau) V + b_{12}(\tau) = 0 \tag{1.11}$$

Law of velocity transformation. Let $dx/dt = V''$ be the particle velocity, measured by a fixed observer, and let $dx'/dt' = V'$ be the particle velocity in a moving frame of reference. Then by (1.9) and (1.10),

$$V'' = \frac{b_{11}(-\tau) V' + b_{12}(-\tau)}{b_{21}(-\tau) V' + b_{22}(-\tau)} = \frac{b_{12}(\tau) - b_{22}(\tau) V'}{b_{21}(\tau) V' - b_{11}(\tau)} \tag{1.12}$$

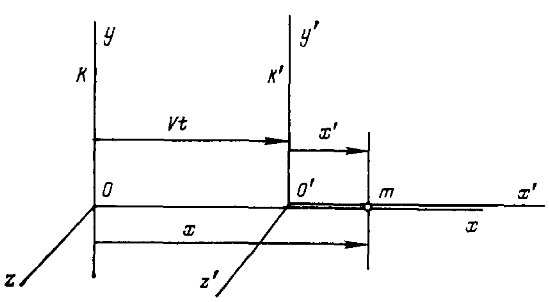

Fig.1

Variation factors of geometric scales and clock behaviour in inertial systems. We fix the points x^1, x^2 and the distance $\Delta x = x^1 - x^2 = l$ in the fixed system K and find the value of Δx measured by a moving observer in K' with $t_1' = t_2'$; on again using (1.9) and (1.10), we find

$$l' = \Delta x' = \lambda_x l,\ \lambda_x = b_{11}{}^{-1}(-\tau) = [b_{11}(\tau) b_{22}(\tau) - b_{12}(\tau) b_{21}(\tau)]/b_{22}(\tau) \tag{1.13}$$

We fix the clocks at the point x'^1, x'^2 of the moving system K'. Let their readings be t_1', t_2'. We find the corresponding readings $t_1, t_2, \Delta t = t_2 - t_1$, of the clocks in the fixed system K. By (1.9) and (1.10),

$$\Delta t' = \lambda_t \Delta t,\ \lambda_t = b_{22}^{-1}(-\tau) = |b_{11}(\tau) b_{22}(\tau) - b_{12}(\tau) b_{21}(\tau)|/b_{11}(\tau) \tag{1.14}$$

Relativistic analogues of Newton's second law. A Lagrangian description can be given of the mechanics controlled by groups of motions of a Riemann space. The construction of the dynamic law in this situation presents no difficulties. In other cases, other arguments must be used to construct the force transformation law, but we shall not dwell on these. But they are not applicable in one-dimensional dynamics. It is therefore natural to confine ourselves, as Zhuravlev did previously (see Foundations of Mechanics. Procedural Aspects. Preprint 251, Institute of problems in Mechanics, Academy of Sciences of the USSR, Moscow, 1985), to the simplest hypothesis which is satisfied in one-dimensional classical mechanics (force is an invariant).

We give an analysis below only for the moment interest groups /10/.

2. ***Groups*** G^{c_0}. When Eqs.(1.2) and (1.4) are satisfied, these groups are distinguished by the conditions

$$\begin{gathered} f_1 \equiv c_{12}{}^2 + c_{13}{}^3 = 0,\ f_2 \equiv c_{13}{}^1 + c_{23}{}^2 \neq 0,\ f_3 \equiv c_{12}{}^1 - c_{23}{}^3 = 0 \\ \psi_2/f_2{}^2 \equiv (\theta_2{}^2 + 4uv)/f_2{}^2 = c_0,\ \theta_2 \equiv c_{13}{}^1 - c_{23}{}^2,\ u \equiv c_{23}{}^1,\ v \equiv c_{13}{}^2 \end{gathered} \tag{2.1}$$

The parameter c_0 takes all real values apart from the limiting values $0, \pm\infty$. We can rewrite (2.1) as $D = (c_0 - 1) f_2{}^2/4$.

Using this equation, we find the roots of the characteristic equation of system (1.8)

$$\lambda_{1,2} = {}^1/_2\,(1 \pm \sqrt{c_0}) f_2$$

The nature of our future expressions will depend on the sign of the parameter c_0. We will confine ourselves to the case $c_0 = \alpha^2$ $(\alpha > 0)$.

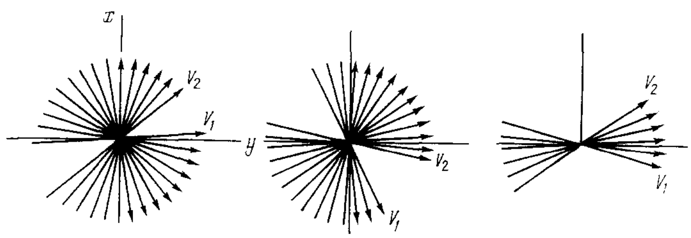

Fig.2

Transformation of the geometric coordinate and time by (1.9) gives

$$\begin{gathered} x' = \frac{x}{\alpha f_2}(A_1 e^{\lambda_1 \tau} + A_2 e^{\lambda_2 \tau}) + \frac{c_{23}{}^1 t}{\alpha f_2}(e^{\lambda_1 \tau} - e^{\lambda_2 \tau}) \\ t' = \frac{x}{\alpha f_2 c_{23}{}^1} A_1 A_2 (e^{\lambda_1 \tau} - e^{\lambda_2 \tau}) + \frac{t}{\alpha f_2}(A_1 e^{\lambda_2 \tau} + A_2 e^{\lambda_1 \tau}) \\ A_{1,2} = {}^1/_2\,(\alpha f_2 + \theta_2) \end{gathered} \tag{2.2}$$

The connection (1.11) between the group parameter τ and the velocity V is given by

$$e^{(\lambda_1 - \lambda_2)\tau} = (c_{23}{}^1 - A_2 V)/(c_{23}{}^1 + A_1 V) \equiv Q$$

with the aid of which we can write transformation (2.2) as

$$\begin{gathered} x' = c_{23}{}^1 (x - Vt) R^{-1} Q^{1/(2\alpha)},\quad R = \sqrt{(c_{23}{}^1 + A_1 V)(c_{23}{}^1 - A_2 V)} \\ t' = c_{23}{}^1 [-c_{13}{}^2 V x + (c_{23}{}^1 + \theta_2 V) t] R^{-1} Q^{1/(2\alpha)} \end{gathered} \tag{2.3}$$

Here we must have $c_{23}{}^1 > 0$. In future we shall confine ourselves to the most interesting siutations of general position when $c_{13}{}^2 \neq 0$, and the group structural parameter α is in no way specialized. It is clear from (2.1) that in these cases

$$A_1 A_2 = {}^1/_4\,(\alpha^2 f_2{}^2 - \theta_2{}^2) = c_{23}{}^1 c_{13}{}^2 \neq 0$$

Let V_1 be the lesser and V_2 the greater of the numbers $c_{23}{}^1/A_2$, $-c_{23}{}^1/A_1$. Relations (2.3) will then retain their meaning if the velocity V satisfies the conditions

$$V < V_1 \text{ or } \quad V > V_2 \text{ for } V_1V_2 > 0$$
$$V_1 < V < V_2 \text{ for } V_1V_2 < 0 \tag{2.4}$$

The domains that correspond to admissible values of V are filled by world lines in Fig.2. By (1.12), the velocity transformation law is

$$V'' = [c_{23}{}^1 (V + V') + \theta_2 VV']/(c_{23}{}^1 + c_{13}{}^2 VV') \tag{2.5}$$

It is independent of the parameter α.

The quantities V_i represent the genus of the limiting velocities of motion, which are the same in any inertial frame of reference, and in this sense can be regarded as signal propagation velocities. In fact, putting $V'' = V'$ in (2.5), we obtain

$$V'_{1,2} = (\theta_2 \pm \alpha f_2)/(2c_{13}{}^2)$$
$$\frac{\theta_2 + \alpha f_2}{2c_{13}{}^2} = \frac{2c_{23}{}^1}{\alpha f_2 - \theta_2} = \frac{c_{23}{}^1}{A_2}, \quad \frac{\theta_2 - \alpha f_2}{2c_{13}{}^2} = -\frac{2c_{23}{}^1}{\alpha f_2 + \theta_2} = -\frac{c_{23}{}^1}{A_1}$$

From (1.13) and (1.14), we find that the variation factors of geometric scales and clock behaviour are

$$\lambda_x = RQ^{1/(2\alpha)}/(c_{23}{}^1 + \theta_2 V), \quad \lambda_t = RQ^{1/(2\alpha)}/c_{23}{}^1 \tag{2.6}$$

Comparison of (1.6), (1.7) and (2.1) shows that, if $c_0 \neq 1$, the groups G^{c_0} do not preserve a metric; if $c_0 = 1$, the preserved metric is degenerate. When finding generalizations of Newton's second law, we regard force, like mass, as an invariant.

The operators $X_k\ (k = 1, 2, 3)$, continued in velocity and acceleration, are

$$X_1{}^* = X_1 = \partial/\partial x, \quad X_2{}^* = X_2 = \partial/\partial t$$
$$X_3{}^* = X_3 + (-c_{13}{}^2 \dot{x}^2 + \theta_2 \dot{x} + c_{23}{}^1)\,\partial/\partial \dot{x} + (-3c_{13}{}^2 \dot{x} + \theta_2 - c_{23}{}^2)\,\ddot{x}\,\partial/\partial \ddot{x}$$

The unique invariant of the continued group, dependent on x'', is

$$X_k{}^*\Omega = 0 \quad k = 1, 2, 3$$

Assuming that it is linear in the acceleration, we obtain

$$\Omega = \ddot{x}\,(1 - A_2\dot{x}/c_{23}{}^1)^{-3/2-1/(2\alpha)}(1 + A_1\dot{x}/c_{23}{}^1)^{-3/2+1/(2\alpha)}$$

The required law is

$$m\ddot{x} = \left[\left(1 - \frac{A_2}{c_{23}{}^1}\,\dot{x}\right)\left(1 + \frac{A_1}{c_{23}{}^1}\,\dot{x}\right)\right]^{3/2}\left(\frac{1 - A_2\dot{x}/c_{23}{}^1}{1 + A_1\dot{x}/c_{23}{}^1}\right)^{1/(2\alpha)} \tag{2.7}$$

Obviously, the domains of admissible values of the velocity x' are the same as in the transformation law (2.3) and relations (2.4).

3. *The Lorentz group.* The Lorentz group (G^3) is distinguished by the conditions

$$f_k = 0, \quad \psi_2 = 4D > 0 \tag{3.1}$$

The transformation of space-time is obtained by solving the Cauchy problem (1.1)

$$x' = \{x\,[(\lambda + c_{13}{}^1)\,e^{\lambda\tau} + (\lambda - c_{13}{}^1)\,e^{-\lambda\tau}] + c_{23}{}^1 t\,(e^{\lambda\tau} - e^{-\lambda\tau})\}/(2\lambda)$$
$$t' = \{x c_{13}{}^2\,(e^{\lambda\tau} - e^{-\lambda\tau}) + t\,[(\lambda - c_{13}{}^1)\,e^{\lambda\tau} + (\lambda + c_{13}{}^1)\,e^{-\lambda\tau}]\}/(2\lambda)$$

In accordance with conditions (3.1), the dependence of the parameter on the velocity V is

$$e^{2\lambda\tau} = [c_{23}{}^1 + (c_{13}{}^1 - \lambda)\,V]/[c_{23}{}^1 + (c_{13}{}^1 + \lambda)\,V], \quad \lambda^2 = D > 0$$

Hence

$$x' = c_{23}{}^1\,(x - Vt)\,R_1^{-1},\ t' = [-c_{13}{}^2 Vx + (c_{23}{}^1 + 2c_{13}{}^1 V)\,t]\,R_1^{-1}$$
$$R_1 = \sqrt{(c_{23}{}^1 + (\lambda + c_{13}{}^1)\,V)(c_{23}{}^1 + (c_{13}{}^1 - \lambda)\,V)} \tag{3.2}$$

(The same result can be obtained from the usual Lorentz transformation by affine replacement in the image and pre-image. But then, instead of the structural constants, the coefficients of the affine transformation appear in the answer).

As in the previous case, the velocity transformation law is described by (2.5) $(\theta_2 = 2c_{13}{}^1)$.

The signal propagation velocities (velocities of motions which are invariant under the choice of inertial system)

$$(c_{13}^1 + \lambda)/c_{13}^2 = c_{23}^1/(\lambda - c_{13}^1), \quad (c_{13}^1 - \lambda)/c_{13}^2 = -c_{23}^1/(c_{13}^1 + \lambda)$$

are the same as the values of V which cause the factors under the radical in (3.2) to vanish. The domains of admissible velocities are qualitatively the same as shown in Fig.2.

The variation factors of the geometric scales and clock behaviour in the inertial systems are

$$\lambda_x = R_1/(c_{23}^1 + 2c_{13}^1 V), \quad \lambda_t = R_1/c_{23}^1 \tag{3.3}$$

In the case of most interest when $c_{23}^1 c_{13}^2 > 0$, we have

$$V_1 < 0, \quad V_2 > 0 \tag{3.4}$$

In the domains (3.4) of admissible values of V, the functions (3.3) have no singularities.

The Lorentz group preserves an indefinite metric $(D > 0)$. The dynamic law is obtained by the classical method

$$mx^{\cdot\cdot} = (1 - 2c_{23}^2 x^{\cdot}/c_{23}^1 - c_{13}^2 x^{\cdot 2}/c_{23}^1)^{3/2} \tag{3.5}$$

4. ***Remarks on coordinate forms of transformations and the position of the groups*** $G^{c_0}(c_0 > 0)$ ***in the general scheme.*** We have obtained in the last two sections relations which characterize the most interesting mechanics of one-dimensional motions of a particle, from which Newtonian mechanics is obtained by a passage to the limit.

These relations contain two or three arbitrary constants, which are combinations of the structural constants of the relevant group.

If no physical arguments can be given, whereby the number of constants can be reduced, they all have to be regarded as world constants of two-dimensional space-time. (The particle dynamics will naturally depend on them).

Obviously, in the context of the present statement of the problem, when we can operator with only four types of object (Newton's second law, the principle of invariance, the set of transformation groups, and the condition for homogeneity of space-time), all the possibilities have been exhausted during the above constructions.

The number of constants might be reduced by a suitable change of variables

$$x_1 = dx + et, \ t_1 = gx + ht, \ dh - ge \neq 0 \tag{4.1}$$

which preserve the homogeneity of space-time (which is equivalent to replacing the basis in the subalgebra $\{X_1, X_2\}$). However, in the case of purely mathematical operation of transition (4.1) to a fixed coordinate form, foundations are still needed for regarding $x_1, t_1,$ and not x, t, as physical variables.

These reasons only appear when we impose the extra requirement that Maxwell's equations be invariant in vacuo under space-time transformations. It can be shown that transformations of the groups G^1, G^7 and $G^{c_0}\ (c_0 < 0)$ of /10/ do not satisfy this requirement even approximately for any coordinate form. The mechanics corresponding to these groups are in contradiction to electrodynamics and have therefore been discarded.

The situation is different for groups $G^{c_0}\ (c_0 > 0)$ and the Lorentz group (G^3). We shall speak about groups G^{c_0} in the next remark. To explain the role of coordinate forms, we shall only dwell on the Lorentz transformations. With

$$c_{13}^1 = 0, \ c_{23}^1/\lambda = c, \ c_{13}^2/c_{23}^1 = c^{-2}$$

relations (3.2) becomes Lorentz transformations, written in the usual coordinate form

$$x' = (x - Vt)\,\beta, \ t' = (-xV/c^2 + t)\,\beta, \ \beta = [1 - (V/c)^2]^{-1/2} \tag{4.2}$$

With

$$c_{13}^1 \neq 0, \quad c_{23}^1 c_{13}^2 > 0$$

in accordance with the law of velocity transformations (2.5), the propagation of light (a signal) is different in different directions: $|V_1| \neq |V_2|$, which can be interpreted as meaning that the space is not isotropic.

Note that this assumption does not contradict experiment, since the error with which the velocity of light is measured at the present time is still computed as several hundred metres per second.

The fact that the velocity of light may depend on the direction of propagation was assumed by Poincaré, Einstein, and Reichenbach, in connection with the problem of measuring time and the analysis of the simultaneity of events /11/. The dependence on the direction of the coordinate velocity of light has been discussed in detail in /11/, whereas the physical velocity of light has been regarded as constant.

Let us return to groups $G^{c_0}\ (c_0 = \alpha^2, \alpha > 0)$. It is clear from conditions (2.1) that, if $\theta_2 = 0\ (c_{23}^1 > 0)$, then $c_{13}^2/c_{23}^1 = (\alpha f_2/(2c_{23}^1))^2$. Assuming for clarity that $\alpha f_2/(2c_{23}^1) > 0$, under the conditions

$$\theta_2 = 0, \quad c_{13}{}^2/c_{23}{}^1 = c^{-2}, \quad \alpha f_2/(2c_{23}{}^1) = c^{-1} \tag{4.3}$$

(c is the velocity of light), we find from transformation (2.3) that

$$x' = (x - Vt)\beta\chi, \quad t' = (-xV/c^2 + t)\beta\chi$$
$$\chi = [(1 - V/c)/(1 + V/c)]^{1/(2\alpha)} \tag{4.4}$$

Hence, as $\alpha \to \infty$ $(f_2 \to 0)$, we obtain the Lorentz transformation (4.2). The dynamic law (2.7) then becomes the law (3.5).

Let us consider in more detail transformations (4.4) of groups G^{c_0} and some of their corollaries, assuming that the values of the parameter $c_0 = \alpha^2$ $(\alpha > 0)$ are sufficiently large.

Under conditons (4.3), relations (2.6) become

$$\lambda_x = \lambda_t = \beta\chi$$

These functions are shown by the broken curve of Fig.3. The continuous curve is the similar graph for the Lorentz group.

If we pass to four-dimensional space-time, by defining the transformation corresponding to (4.4) of the other two coordinates by

$$y' = y\chi, \quad z' = z\chi \tag{4.5}$$

the following can be seen. As distinct from Lorentz transformations, (4.4) and (4.5) do not preserve the pseudo-Euclidean metric $ds'^2 = \chi^2 ds^2$, but preserve the light cone

$$ds^2 \equiv dt^2 - (dx^2 + dy^2 + dz^2)/c^2 = 0$$

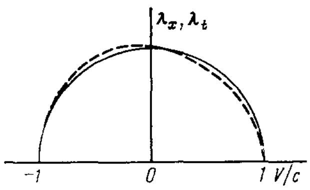

Fig.3

Consequently, Maxwell's equations are invariant under these transformations. (In the four-dimensional case, the parameter α defines the similitude factor). In short, these transformations differ by as little as desired from the Lorentz transformation (for large α), admit of a passage to the limit to the Galilean (as $c \to \infty$), and preserve Maxwell's equations.

There are considerable qualitative differences between the properties of the relevant space-time continua, no matter how close the properties are quantitatively. The differences lie in the dependence of the properties of the scales and clock behaviour on the sign of their velicity of motion with respect to a fixed observer (violation of parity) and the dependence of the body cross-sections on the velocity (see (4.7)).

5. *Particle inertial motions controlled by Galilean, Lorentz, and conformal transformation groups.* In the previous sections we have mainly been concerned with one-dimensional dynamic models. It has been said that the three-dimensional models of motion, controlled by groups of dimensionality 10 (or higher) are more interesting. In the present section, it is in this full-dimensional statement that we consider one of the primary problems of particle dynamics, namely, the construction of the inertial motions that are controlled, in the sense of the invariance principles /1, 2/, by the most interesting transformation groups. We propose a method of solving such problems. It is applicable to the groups of Galilean, Lorentz, and conformal, transformations. We will describe in detail the properties of the conformally invariant particle motions in both the sub- and super-light domains.

A study of the inertial particle motions is of the first importance for two reasons: first, these motions are at the basis of the invariant definition of inertial reference systems, are second, they serve as an important guide for constructing dynamic laws.

Formulation of the problem and the result. By a law of particle motion we shall mean, as usual, the time dependence of the particle coordinates and some set of initial conditions. If the number of conditions is large enough, the given transformations will preserve a wide class of motions. If, on the other hand, there are too few conditions, there may be no motions at all which are preserved by the group. The cases when it is possible to introduce just as many initial conditions as are needed to obtain a unique law of motion are interesting; the concreteness of the law strengthens the confidence in its likelihood and facilitaties its experimental verification.

We shall seek the motions which depend on the initial position, velocity, and acceleration of the particle (this is stipulated by the number of parameters (15) of the widest of the transformation groups, namely, the conformal ones, which ensures a unique law of inertia)

$$x_k = \varphi_k\,(t,\, t_0;\, x_1{}^\circ,\, x_2{}^\circ,\, x_3{}^\circ;\, v_1,\, v_2,\, v_3;\, w_1,\, w_2,\, w_3) \tag{5.1}$$

which is transformed into itself

$$x_k' = \varphi_k\,(t',\ t_0':\ x_1'^\circ,\ x_2'^\circ,\ x_3'^\circ;\ v_1,\ v_2,\ v_3;\ w_1,\ w_2,\ w_3)$$

by all the transformations of the group. Here, x_k are the Cartesian coordinates of the particle, and v_k, w_k are the projections of its velocity and acceleration at the initial instant t_0.

The method given below leads to the following result.

The most general laws of particle motion of this class, which are invariant under Galilean, Lorentz, and conformal, transformations, are

$$x_k = x_k^\circ + v_k t^* + b w_k, \quad t^* = t - t_0 \tag{5.2}$$

For the Galilean group, b is an arbitrary function $t^*, w = (\sum_k w_k^2)^{1/2}$.

For the Lorentz group, b is a function of the variables $\omega_1 = \sum_k v_k^2 = v^2$, $\omega_2 = \sum_k w_k^2 = w^2$, $\varepsilon = \sum_k v_k w_k$, which satisfies the equations

$$\begin{gathered}2\omega_1^* \frac{\partial b}{\partial \omega_1^*} + 3\varepsilon \frac{\partial b}{\partial \varepsilon} + 4\omega_2 \frac{\partial b}{\partial \omega_2} - t^* \frac{\partial b}{\partial t^*} + 2b = 0\\ \omega_1^* \frac{\partial b}{\partial \varepsilon} + 2\varepsilon \frac{\partial b}{\partial \omega_2} - b \frac{\partial b}{\partial t^*} = 0, \quad \omega_1^* = v^2 - c^2\end{gathered} \tag{5.3}$$

(c is the velocity of light).

For the conformal group

$$\begin{gathered}b = (-\varepsilon t^* + \omega_1^* \pm \sqrt{R_0})\, w^{-2}\\ R_0 = (\omega_1^* - \varepsilon t^*)^2 - \omega_1^* w^2 t^{*2} = (\varepsilon^2 - \omega_1^* w^2)\, t^{*2} - 2\varepsilon\omega_1^* t^* + \omega_1^{*2}\end{gathered} \tag{5.4}$$

In (5.4), we have to put the lower sign in front of the radical when $\omega_1^* > 0$, and the upper sign when $\omega_1^* < 0$.

The conformally invariant motions satisfy the vector differential equation

$$d\mathbf{W}/dt = -3\,(\mathbf{V}, \mathbf{W})(c^2 - \mathbf{V}^2)^{-1}\,\mathbf{W} \tag{5.5}$$

where $\mathbf{V}$ and $\mathbf{W}$ are the velocity and acceleration at the current instant, and $(\mathbf{V}, \mathbf{W})$ is their scalar product.

We omit the proofs of (5.2)-(5.5).

Corollary. 1°. There are no invariant particle motions, dependent only on the initial position.

2°. The conformally invariant motions uniquely define the initial positions, velocity, and acceleration.

3°. When there is no dependence on the initial acceleration ($b \equiv 0$) the motions which are invariant under transformations of the Galilean and Lorentz groups are uniquely defined. This law of inertia is Galilean.

The producedure for solving the problem. In accordance with the statement of the problem, the required law of motion (5.1) describes the invariant manifold of the given transformation group, continued onto the chosen set of initial data.

The invariant manifolds are best evaluated by using the Lie algebra instead of the group (/12/, p.178).

The right-hand sides φ_k of Eqs.(5.1) are evaluated from conditions

$$\begin{gathered}X_l^*\Phi_k|_\Gamma = 0, \quad \Phi_k \equiv x_k - \varphi_k, \quad \Gamma: \Phi_k = 0\\ l = 1, \ldots, n; \quad k = 1, 2, 3\end{gathered} \tag{5.6}$$

Each of the basis operators X^* of the continued algebra can be expressed in terms of the corresponding operators of the basic algebra

$$X = \xi_k \partial/\partial x_k + \xi \partial/\partial t$$

by the relations

$$X^* = X + \xi_k^\circ \frac{\partial}{\partial x_k^\circ} + \xi^\circ \frac{\partial}{\partial t_0} + \eta_k^\circ \frac{\partial}{\partial v_k} + \zeta_k^\circ \frac{\partial}{\partial w_k}$$

$$\eta_k = \frac{d\xi_k}{dt} - v_k \frac{d\xi}{dt}, \quad \zeta_k = \frac{d\eta_k}{dt} - w_k \frac{d\xi}{dt}$$

(ξ_k°, ξ°, η_k°, ζ_k° are the result of substituting the initial conditions into the functions ξ_k, ξ, η_k, ζ_k

Calculations using (5.6) present no essential difficulties and lead to the required result. This method has advantages over others: it is applicable for any transformation groups (and not just for groups of motions of a Riemann space) which are given by their Lie algebra; it gives the law of motion in the final (integrated) form, and it does not require an explicit knowledge of the final transformations of the group.

Kinematics of conformally invariant particle motions. Invariance under conformal transformations, among which are included Lorentz transformations, plays an important role and has been used more than one in physics, see e.g., /13/. This is becuase conformal transformations preserve Maxwell's equations in vacuo /14/ so that they must somehow be connected with the fundamental properties of space-time, and notably, with the law of inertia.

Let us list some general properties of conformally invariant motions.

1°. The motions are performed with both sub- and super-light velocities.

2°. If the particle initial velocity and acceleration are collinear, $\mathbf{w} = \lambda \mathbf{v}$, the trajectories of motion in geometric space $\{x_1, x_2, x_3\}$ are straight lines. Such motions are called relativistic uniformly accelerated motions.

Fig.4

Fig.5

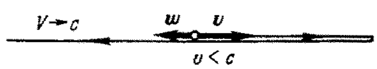

Fig.6

Fig.7

Fig.8

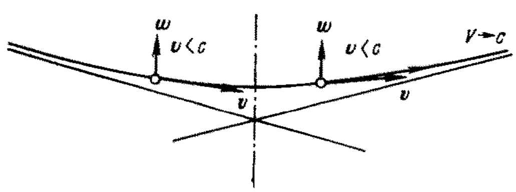

Fig.9

3°. If the initial velocity and acceleration are not collinear ($\mathbf{w} \neq \lambda \mathbf{v}$), the trajectories in the gometric space are plane second-order curves, which lie in the plane defined by the vectors $\mathbf{v}$ and $\mathbf{w}$; in the sublight domain they are unbounded arcs of hyperbola, while in the superlight domain they are unbounded or finite arcs of a hyperbola, parabola, or ellipse. (The sublight motions were given earlier in /15/).

4°. Depending on the initial conditions, the particle can either move away to infinity with velocity $V \to c$ as $t \to \infty$, or else the motion terminates in a finite time at a finite point of space with velocity $V \to \infty$ (this can only occur if $v > c$).

5°. Given the initial velocity $v > c$, the type of trajectory and the kind of motion are completely defined the angle γ between the initial velocity and acceleration vectors, i.e., they depend on the relation between γ and the critical angle γ_0, where

$$\cos \gamma_0 = \sqrt{1 - c^2/v^2}, \quad 0 < \gamma_0 < \pi/2$$

Let us describe typical situations in more detail.

1) $w = 0$. Uniform rectilinear motion with constant velocity $V = v$ $(v \lessgtr c)$ (Fig.4)

2) $\mathbf{w} = \lambda\mathbf{v}$, $0 < v < c$, $\lambda > 0$. The motion is rectilinear, the velocity V increases monotonically, $\lim V = c$ as $t^* \to \infty$ (Fig.5).

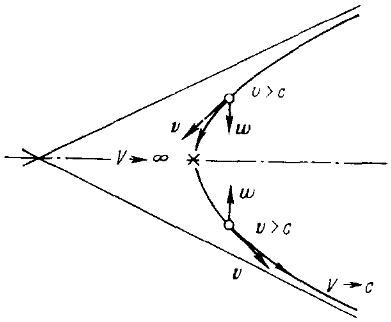

Fig.10

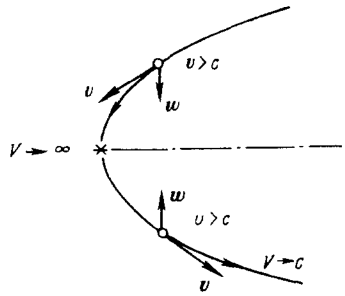

Fig.11

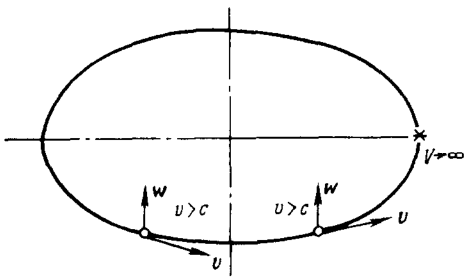

Fig.12

3) $\mathbf{w} = \lambda\mathbf{v}$, $0 < v < c$, $\lambda < 0$. The motion is rectilinear. The velocity decreases to zero, then, changing sign, it increases monotonically, $\lim V = c$ as $t^* \to \infty$ (Fig.6).

4) $\mathbf{w} = \lambda\mathbf{v}$, $v > c$, $\lambda > 0$. The motion is rectilinear. It breaks off at the instant $t_*^* = \omega_1^* (\varepsilon + w\sqrt{\omega_1^*})^{-1}$ at a finite point of space. The velocity increases monotonically, $\lim V = \infty$ as $t^* \to t_*^*$ (Fig.7).

5) $\mathbf{w} = \lambda\mathbf{v}$, $v > c$, $\lambda < 0$. The motion is rectilinear. The velocity decrease monotonically, $\lim V = c$ as $t^* \to \infty$ (Fig.8).

6) $v < c$, $\varepsilon = (v, w) > 0$. The trajectory is a branch of a hyperbola which does not contain the vertex. On moving away, the particle approaches the asymptote without limit. The velocity increases monotonically, $\lim V = c$ as $t^* \to \infty$ (Fig.9, right-hand point).

7) $v < c$, $(v, w) < 0$. The trajectory is an unbounded arc of a hyperbola. The particle passes through the vertex, then, on moving away, it approaches the asymptote without limit. The velocity decreases, then increases monotonically, $\lim V = c$ as $t^* \to \infty$ (Fig.9, the left-hand point).

8) $v > c$, $-\gamma_0 < \gamma < \gamma_0$. The motion is over a bounded arc of a hyperbola with monotonically increasing velocity, then breaks off at the instant $t^* = t_*^*$ of reaching the vertex, $\lim V = \infty$, as $t^* \to t_*^*$ (Fig.10, the upper point).

9) $v > c$, $\pi - \gamma_0 < \gamma < \pi + \gamma_0$. The motion is over an unbounded arc of a hyperbola from the vertex with monotonically decreasing velocity $\lim V = c$ as $t^* \to \infty$ (Fig.10, the lower point).

10) $v > c$, $\gamma = \gamma_0$, or $\gamma = 2\pi - \gamma_0$. The motion is over a bounded arc of a parabola with monotonically increasing velocity, and breaks off at the instant $t^* = t_*^*$ of reaching the vertex, $\lim V = \infty$ as $t_* \to t_*^*$ (Fig.11, the upper point).

11) $v > c$, $\gamma = \pi - \gamma_0$, $\gamma = \pi + \gamma_0$. The motion is over the branch of a parabola from the vertex with monotonically decreasing velocity, $\lim V = c$ as $t^* \to \infty$ (Fig.11, the lower point).

12) $v > c$, $\pi/2 < \gamma < \pi - \gamma_0$, $\pi + \gamma_0 < \gamma < 3\pi/2$. The motion is over an arc of an ellipse. The particle approaches the minor semi-axis with decreasing velocity, and after intersecting it the motion continues in the previous direction with monotonically increasing velocity. The motion stops at the instant t_*^* of intersecting the major semi-axis, $\lim V = \infty$ as $t^* \to t_*^*$ (Fig.12, the left-hand point).

13) $v > c$, $\gamma_0 < \gamma \leqslant \pi/2$, $(3/2)\pi \leqslant \gamma < 2\pi - \gamma_0$. The motion is over the arc of an ellipse from its intersection with the minor semi-axis with monotonically increasing velocity. The motion stops at the instant $t^* = t_*^*$ of intersecting the major semi-axis, $\lim V = \infty$ as $t^* \to t_*^*$ (Fig.12, the right-hand point).

On proving the properties of conformally invariant motions. The trajectories. We take the point $\{x_1^\circ, x_2^\circ, x_3^\circ\}$ as the origin, the Ox axis along the initial acceleration w, and the Oy axis in the plane of the vectors v, w (if $\mathbf{w} = \lambda\mathbf{v}$, the trajectory is obviously a straight line). We obtain

$$x_1 = (\omega_1^* \pm \sqrt{R_0})/w, \quad x_2 = v_2 t^*, \quad x_3 = 0 \tag{5.7}$$

Hence

$$v_2^2 x_1^{*2} - (\varepsilon^2 - \omega_1^* w^2) x_2^{*2} = B \text{ when } \varepsilon^2 - \omega_1^* w^2 \neq 0 \tag{5.8}$$

$$x_1^* = w x_1 - \omega_1^*, \quad x_2^* = x_2 - \varepsilon\omega_1^* v_2/(\varepsilon^2 - \omega_1^* w^2)$$

$$B = -\omega_1^{*3} v_2^2 w^2/(\varepsilon^2 - \omega_1^* w^2)$$

$$v_2^2 x_1^{*2} = -2\varepsilon\omega_1^* v_2 x_2 + \omega_1^{*2} v_2^2 \text{ when } \varepsilon^2 - \omega_1^* w^2 = 0 \tag{5.9}$$

The types of trajectory depends on the sign of $c^2 - v^2 \sin\gamma$, or what amounts to the same thing, on the sign of $\varepsilon^2 - \omega_1^* w^2$ (γ is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$): if the latter expression is greater than zero, the trajectory is a hyperbola, if it is equal to zero, the trajectory is a parabola, and if less than zero, it is an ellipse.

Using Eqs.(5.7)-(5.9), we can find the point on the trajectory at which the motion stops in the superlight domain, given suitable initial conditions, and also the disposition of the trajectory relative to the direction of the initial acceleration. For this, we only need to note that the motion stops when R_0 vanishes, i.e., at the point of intersection of the trajectory with the Ox_2^* axis, and that the vector w is along the Ox_1^* axis.

The initial positions of the particle on the trajectory and its subsequent motion are easily found from relations for the running velocity and acceleration which follows from (5.2):

$$V_i = v_i - \varepsilon w_i w^{-2} + w_i w^{-2} R_0^{-1/2} [\omega_1^*\varepsilon + (\omega_1^* w^2 - \varepsilon^2) t^*] \operatorname{sgn} \omega_1^*$$

$$V^2 = c^2 + \omega_1^{*3} R_0^{-1}, \quad W_i = |\omega_1^*|^3 w_i R_0^{-3/2}, \quad W^2 = w^2 \omega_1^{*6} R_0^{-3}$$

$$(\mathbf{V}, \mathbf{W}) = \sum_k V_k W_k = |\omega_1^*|^3 [\omega_1^*\varepsilon + (\omega_1^* w^2 - \varepsilon^2) t^*] R_0^{-2}$$

and from the radius of curature ρ of the trajectory for the running position of the particle

$$\rho = \rho_0 v^{-3} (\omega_1^* + c^2 \omega_1^{*-3} R_0)^{3/2}$$

It is important to take account of the following factors here:

1) the particle position and its velocity and acceleration are defined only for $R_0 > 0$;

2) if $v < c$ we have $R_0 > 0$ for all $t^* \in]-\infty, \infty[$;

3) if $v > c$, then R_0 has two zeros:

$$t_1^* = \omega_1^* (\varepsilon + w\sqrt{\omega_1^*})^{-1}, \quad t_2^* = \omega_1^* (\varepsilon - w\sqrt{\omega_1^*})^{-1}$$

which are either both positive or both negative, if $\varepsilon^2 - \omega_1^* w^2 > 0$; while they are of different signs if $\varepsilon^2 - \omega_1^* w^2 < 0$; if $\varepsilon^2 - \omega_1^* w^2 = 0$ the function R_0 has one zero $t^* = \omega_1^*/(2\varepsilon)$;

4) the particle accelerations remain parallel to one another at any positions on the trajectory. Their common direction is the same as that of the axis of symmetry of the trajectory in the sublight domain, and perpendicular to it in the superlight domain. If the trajectory is an ellipse, the acceleration is directed along the minor axis;

5) in the superlight domain the motion can only terminate at the vertex of the trajectory;

6) the radius of curvature of the trajectory increases or decreases along with R_0.

To sum up, by using Klein's approach, based on a preliminary sampling of the groups and a study of their hierarchies, we can give a uniform logical treatment of the problem of constructing mechanical models and arrive at expressions which can be checked experimentally. It is thus worth gaining experience in constructing models similar to those in the present paper.

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THE ENTRY OF A WEDGE INTO AN INCOMPRESSIBLE FLUID*

B.S. CHEKIN

The similarity problem of the entry of a rigid wedge into an ideal weightless incompressible fluid occupying a half-space is studied. The difficulty is that a non-linear boundary conditon has to be satisfied on the free surface of the fluid, whose position is unknown and has to be found during the solution. Three types of fluid motion are considered: flow past the wedge without break-away, the case when one wedge face is not wetted (a semi-infinite plate), and the intermediate case, when a cavity forms on one face. The problem amounts to solving a non-linear system of integral equations. A method of solving this system is given for the flow without break-away and the plate case. Examples of calculations are given. The results for thin and thick wedges are compared with approximate data.

The penetration of a wedge into a fluid was first studied in /1/. In /2/ the linear problem of normal collision with a water surface was solved. An approximate solution can be found e.g., in /3-5/. In /6/ a solution was obtained for the special case the entry of a wedge into a fluid. In /7/ the problem of normal wedge entry was solved in the exact non-linear statement, and the same problem was considered in /8/. The method below is based on that of /7/.

1. Let the wedge $M_1M_2M_3$ move with constant velocity $\mathbf{V}_w$ (Fig.1) and enter a fluid which occupies the lower half-space $Y \leqslant 0$ at the initial instant $t = 0$ and is at rest. At an instant $t > 0$ the distorted fluid boundary $N_1M_1M_2M_3N_3$ can have the shape shown in

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